NON-STATIONARY AND NON-LINEAR DISPERSIVE MEDIUM AS EXTERNAL FIELD WHICH GENERATES THE SQUEEZED STATES

A.A.Lobashov
State Metrology Institute
Russia, 198005, St.-Petersburg, Moskovsky pr. 19

V.M.Mostepanenko
St.-Petersburg Technological Institute
Russia, 198013, St.-Petersburg, Moskovsky pr. 26

Abstract

The theory of quantum effects in non-linear dielectric media influenced by pumping external field based directly on Maxwell equations is developed. The diagonalisation of Hamiltonian of quantized generated field by the canonical Bogoliubov transformations allowed to obtain the general expressions for the number of created photons and for the degree of squeezing. As an example for the case of plane pumping wave the results are calculated in the zero order of secular perturbation theory on small parameter characterizing the medium non-linearity. The Heisenberg equations of motion are obtained for non-stationary case and commonly used effective Hamiltonian derived from the first principles of quantum electrodynamics.

As it is well known for theoretical description of squeezed states the quantum treatment of light is necessary. Consideration of the medium as classical one supposes some effective interaction of the pumping and generated waves. For such description effective Hamiltonians were commonly used. But the problem of correspondence between the Heisenberg equations which follow from the effective Hamiltonians and the Maxwell equations for quantized electromagnetic field in the medium was not investigated up to now.

The main contents of our paper is to treat the theory of quantized electromagnetic field propagating in the medium with time dependent dielectric properties on the base of Maxwell equations. This problem is quite analogical with the theory of quantum effects in non-stationary external fields [1]. But in our case the role of "external field" is played not by the pumping field itself but by the induced non-stationary dielectrical properties of the medium.

The non-linear medium is described by the tensors of dielectric sensibilities of second, third and higher ranks which determinate the medium polarization produced by the electric field. In the frame of semiclassical theory we shall decompose the whole electromagnetic field into the sum of intensive classical pumping field $E_{pk}(x)$ and generated by the medium quantized field $\hat{E}_k(x)$

$$E_k(x) = E_{pk}(x) + \hat{E}_k(x). \qquad (1)$$

Supposing the pumping field to be more intensive then the generated one we can omit the terms in the operator of electric induction which are higher than linear in quantized field. From the quantized Maxwell equations in the medium the following integro-differential equation follows for the operator of vector-potential $\hat{A}_k(x)$ (we use gauge with $\hat{A}_0 = 0, \partial_k \hat{A}_k(x) = 0$)

$$\frac{\partial}{\partial t}(K_{ij}\frac{\partial}{\partial t}\hat{A}_j)(x) - \Delta \hat{A}_i(x) = 0, \tag{2}$$

where

$$K_{ij} = 1 + L_{ij} + N_{ij}, \tag{3}$$

$$(L_{ij}\hat{E}_j)(x) = 4\pi \int\limits_{-\infty}^{\infty} \chi_{ij}^{(1)}(t-t';\mathrm{x})\hat{\mathrm{E}}_\mathrm{j}(\mathrm{t}',\mathrm{x})\mathrm{dt}', \tag{4}$$

$$\left(N_{ij}\hat{E}_j\right)(x) = 8\pi \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \chi_{ikj}^{(2)}(t-t', t-t'';\mathrm{x})\mathrm{E_{pk}}(\mathrm{t}',\mathrm{x})\hat{\mathrm{E}}_\mathrm{j}(\mathrm{t}'',\mathrm{x})\mathrm{dt}'\mathrm{dt}''. \tag{5}$$

So we are resulting with the problem of quantization in the external field which is included into the kernel of the integral operator K_{ij}.

The ground of secondary quantization is that the quantized field must be decomposed over the complete system of solutions $u_{i\mathbf{p}}^{(\delta)\sigma}(x)$ of the classical equation corresponding to the quantized one 2

$$\hat{A}_j(x) = \sum_{\sigma,\mathbf{p}}[u_{j\mathbf{p}}^{(-)\sigma}(x)a_\sigma(\mathbf{p}) + \mathrm{u}_{\mathrm{j}\mathbf{p}}^{(+)\sigma}(\mathrm{x})\mathrm{a}_\sigma^+(\mathrm{p})], \tag{6}$$

$$[a_\sigma(\mathrm{p}), \mathrm{a}_{\sigma'}^+(\mathrm{p}')] = \delta_{\sigma\sigma'}\delta_{\mathbf{pp}'}. \tag{7}$$

To orthonormalise the set of solutions it is necessary to introduce a scalar product [2]

$$(u,v) = i\int d^3\mathrm{x} \int\limits_{-\infty}^{t} \mathrm{d}\tau \int\limits_{-\infty}^{\infty} \mathrm{d}\tau' \frac{\partial K_{\mathrm{ij}}(\tau, \tau-\tau';\mathrm{x})}{\partial \tau}[\mathrm{u}_\mathrm{i}^*(\tau,\mathrm{x})\frac{\partial}{\partial \tau'}\mathrm{v}_\mathrm{j}(\tau',\mathrm{x}) - \frac{\partial}{\partial \tau'}\mathrm{u}_\mathrm{j}^*(\tau',\mathrm{x})\mathrm{v}_\mathrm{i}(\tau,\mathrm{x})], \tag{8}$$

$$\left(u_{\mathbf{p}}^{(\pm)\sigma}, u_{\mathbf{p}'}^{(\pm)\sigma'}\right) = \mp\delta_{\sigma\sigma'}\delta_{\mathbf{pp}'}. \tag{9}$$

Operators $a_\sigma(\mathrm{p})$, $a_\sigma^+(\mathrm{p})$ annihilate and create free photons in the medium when time tends to infinity in the state with quantum numbers p, σ. When time increases negative- and positive-frequency solutions will be mixed which has the interpretation as particles-antiparticles creation. As the consequence the role of photons in the medium will be played by quasiparticles which creation-annihilation operators $b_\alpha^+(t), b_\alpha(t)$ (here $\alpha = (\sigma, \mathrm{p})$) diagonalize the Hamiltonian of quantized field in a moment t and which are connected with a_α^+, a_α by the canonical Bogoliubov transformation

$$a_\alpha = \sum_\beta [\Phi_{\alpha\beta}(t)b_\beta(t) + \Psi_{\alpha\beta}(t)b_\beta^+(t)]. \tag{10}$$

The number of the quasiparticles created by the medium in α-state is

$$N_\alpha(t) = < 0_{-\infty}|b_\alpha^+(t)b_\alpha(t)|0_{-\infty} > = \sum_\gamma \Psi_{\alpha\gamma}^+(t)\Psi_{\gamma\alpha}(t). \tag{11}$$

The degree of squeezing is defined by the value of dispersion of quadrature components

$$X_{1\alpha}(t) = \frac{1}{2}\left(b_\alpha^+(t) + b_\alpha(t)\right), X_{2\alpha}(t) = \frac{1}{2i}\left(b_\alpha^+(t) - b_\alpha(t)\right) \tag{12}$$

or by their linear combinations

$$Y_{1,2\beta} = \sum_\alpha Q_{\beta\alpha} X_{1,2\alpha}. \tag{13}$$

The matrix of squeezing is

$$S_{ik,\alpha\beta} =< 0_{-\infty}|Y_{i,\alpha}(t)Y_{k,\beta}(t)|0_{-\infty} >= \frac{1}{4}[Q\,(\Phi \mp \Psi)^+\,(\Phi \mp \Psi)\,Q^T]_{\alpha\beta}, \tag{14}$$

(here minus for $i = k = 1$, plus for $1 = k = 2$).

Let us apply the developed formalism to the quantum process of light generation [3]. We shall suppose the non-linear crystal to be placed in a flat resonator without losses and medium absorption [4]. To obtain solutions of 2 we shall decompose $u_{ip}^{(\delta)\sigma}(x)$ over the space harmonics of resonator. The system of equations for the Fourier coefficients can be solved by the perturbation theory with the small parameter ϵ

$$\epsilon = 8\pi E_{p0} \max_\omega |\frac{\chi^{(2)}(\omega_p, -\omega)}{1 + 4\pi\chi^{(1)}(\omega)}| \ll 1. \tag{15}$$

Because of parametric resonance it is necessary to make use of the secular perturbation theory [5].

From the zero order solution it is easy to obtain the number of created by the medium photons in n-mode

$$N_n(t) = |\theta_n|^2 \sinh^2 \Lambda_n \epsilon t, \tag{16}$$

where θ_n and Λ_n are some constants of the order of 1. From the matrix of squeezing it is seen that the quadrature components dispersions grow exponentially at a large time. However there is a time interval $[0, t_{n\,\min}]$ during which the dispersion of one of the quadrature components is squeezed to the value less than the standard quantum limit $\frac{1}{4}$

$$t_{n\,\min} = \frac{1}{\Lambda_n \epsilon} Arctanh \;\; e^{-r_n}, \quad \sinh r_n = |\sigma_n|, \tag{17}$$

$$S_{nn\,\min} = \frac{1}{4}\frac{2\sinh^2 r_n}{e^{2r_n} - 1} < \frac{1}{4}, \tag{18}$$

where σ_n is proportional to the difference between the sum of generated waves frequencies and the pumping wave frequency caused by medium dispersion. So the frequency upset caused by the medium dispersion destroys squeezing.

As it is commonly known, the diagonalization of Hamiltonian is equivalent to the solution of Heisenberg equations. Now we shall introduce the Heisenberg operators of creation and annihilation and deduce the equations for these operators for the case of non-stationary external field.

Simple differentiation of $b_\alpha(t)$ with the help of Bogoliubov transformation 10 provides that the quasiparticles operators satisfies the following equation

$$\dot{b}_\alpha(t) = \sum_\beta \{[\dot{\Phi}^+(t)\Phi(t) - \dot{\Psi}^T(t)\Psi^*(t)]_{\alpha\beta} b_\beta(t) + [\dot{\Phi}^+(t)\Psi(t) - \dot{\Psi}^T(t)\Phi^*]_{\alpha\beta} b_\beta^+(t)\}. \tag{19}$$

The operators of quasiparticles differ from the Heisenberg operators $c_\alpha(t), c_\alpha^+(t)$ extended to the non-stationary case only by a phase [1]

$$c_\alpha(t) = e^{-i\theta_\alpha(t)} b_\alpha(t), \quad \theta_\alpha(t) = 2\int_{-\infty}^{t} \omega_\alpha(\tau) d\tau, \tag{20}$$

where $\omega_\alpha(\tau)$ is the instant energy of quasiparticle. Remembering that in terms of Heisenberg operators the Hamiltonian is also diagonal and with the help of 19, 20 we obtain the generalized Heisenberg equations

$$\dot{c}_\alpha(t) = -i[c_\alpha(t), H(t)] +$$
$$\sum_\beta [e^{-i\theta_\alpha(t)} \left(\dot{\Phi}^+\Phi - \dot{\Psi}^T\Psi^*\right)_{\alpha\beta} e^{+i\theta_\beta(t)} c_\beta(t) + e^{-i\theta_\alpha(t)} \left(\dot{\Phi}^+\Psi - \dot{\Psi}^T\Phi^*\right)_{\alpha\beta} e^{-i\theta_\beta}(t) c_\beta^+(t)]. \tag{21}$$

In the limits when time tends to infinity Bogoliubov coefficients tend to constants and we are resulting with the ordinary Heisenberg equations.

Inserting the expressions for the Bogoliubov transformation coefficients for zero order perturbation theory into generalized Heisenberg equation 21 we obtain the following equation describing the process of parametric generation of photons in n and $l-n$ modes

$$\dot{c}_n(t) = -i\Omega c_n(t) + \varepsilon\Lambda_n \theta_n^* e^{-i2\Omega t} c_{l-n}^+(t), \tag{22}$$

where Ω is the energy in mode n or $l-n$. It is clearly seen that this equation may be provided as the usual Heisenberg equation $\dot{c}_n = -i[c_n, H]$ by the standard effective Hamiltonian [3]

$$H_{eff}(t) = \Omega c_n^+(t)c_n(t) + \Omega c_{l-n}^+(t)c_{l-n}(t) + \varepsilon\Lambda_n[\theta_n^* e^{-i2\Omega t} c_n^+(t)c_{l-n}^+(t) + \theta_n e^{i2\Omega t} c_n(t)c_{l-n}(t)]. \tag{23}$$

So the standard quadratic effective Hamiltonian is obtained as the zero order of secular perturbation theory applied to the exact integro-differential equation which describes the propagation of quantized electromagnetic field in non-stationary medium. The corrections to it also can be obtained in the frame of exposed here self-consistent description of the process of squeezed states generation based on the first principles of quantum electrodynamics.

References

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